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Connectedness in graded ditopological texture spaces

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Abstract. The aim of this paper is to introduce two different types of connectedness notions for graded ditopological texture spaces: the connectedness function which gives the grade of connectedness of a set and the connectedness spectrum by means of spectrum idea. Also, the properties of these connectedness notions and their relationships with the connectedness notion in ditopological case are investigated. Further, the relation between these two different types of connectedness notions is studied.

1. Introduction and preliminaries

Fuzzy topological spaces were introduced by C. Chang in 1968 [7]. However, in this structure a fuzzy subset is open or not. Then Šostak and Kubiak independently gave a new definition of fuzzy topology where a fuzzy subset has a degree of openness [13, 14, 16].

L. M. Brown has presented ditopological texture spaces as a natural evolvement of [11]. The notion of ditopology is more general than general topology, bitopology and fuzzy topology in Chang's sense. Some essential studies on texture spaces and ditopological texture spaces can be reached from [1–5, 10, 17].

The concept of connectedness has crucial roles for investigating topological structures and so far, they have been studied in numerous topological settings by several authors. Chaudhuri and Das introduced this concept for fuzzy topological spaces in [8], and later it has been presented for ditopological texture spaces in [9].

The theory of graded ditopology has been introduced by Brown and Šostak in [6] and this structure is more comprehensive than both fuzzy topology presented independently by Šostak in [14], Kubiak in [13] and ditopology given in [1, 2]. In this theory, it is not mentioned whether an element of a texture is open (closed) or not. Openness and closedness are rather defined as independent grading functions. So, the theory of graded ditopologies

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provides a different and wider perspective. Yet the generalizations of some properties in the theory of ditopological spaces to this theory are not valid as a matter of course.

In this paper, two different types of connectedness notions for graded ditopological texture spaces are introduced in accordance with the connectedness notion in ditopological texture spaces given in [9]. Firstly, we present the concept of connectedness function which gives the grade of connectedness of a set to us. Afterwards, through spectral theory, as in [12, 15, 16], connectedness spectrum of a set is given. Also, the properties of these connectedness notions and their relationships with the connectedness notion in ditopological case are investigated. Further, the relation between these two different types of connectedness notions is studied. Our basic motivation is to fulfill some missing parts in the theory of graded ditopologies in accordance with the theory of ditopological spaces and comparatively investigate their propersies.

Ditopological Texture Spaces [1, 3, 4]. Let Y be a set and $\mathcal{Y} \subseteq \mathcal{P}(Y)$ with $Y, \emptyset \in \mathcal{Y}$. $\mathcal{Y}$ is called a texturing of Y and $(Y, \mathcal{Y})$ is called a texture space, or simply a texture if the following conditions hold:

(1) $(\mathcal{Y}, \subseteq)$ is a complete lattice which has the property that arbitrary meets coincides with intersections and finite joins coincide with unions.

(2) $\mathcal{Y}$ is completely distributive, i.e., for all index sets I, for all $i \in I$ if J_i is an index set and if $A_i^j \in \mathcal{Y}$ then

$$\bigcap_{i\in I} \bigvee_{j\in J_i} A_i^j = \bigvee_{\gamma\in\Pi_i J_i} \bigcap_{i\in I} A^i_{\gamma(i)}.$$

(3) $\mathcal{Y}$ separates the points of Y, that is, if $y_1, y_2 \in Y$ with $y_1 \neq y_2$ then there exists $A \in \mathcal{Y}$ such that $y_1 \in A$, $y_2 \notin A$ or $y_2 \in A$, $y_1 \notin A$.

In general, a texturing of Y may not be closed under set complementation. However, if there is a mapping $\sigma : \mathcal{Y} \to \mathcal{Y}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{Y}$ then σ is called a complementation on $(Y, \mathcal{Y})$ and $(Y, \mathcal{Y}, \sigma)$ is called a complemented texture.

The $p-sets$ given by $P_y = \bigcap\{A \in \mathcal{Y} \mid y \in A\}$ and the $q-sets$ given by $Q_y = \bigvee\{A \in \mathcal{Y} \mid y \notin A\} = \bigvee\{P_u \mid u \in Y, y \notin P_u\}$ are essential to define several concepts in a texture space $(Y, \mathcal{Y})$.

A texture $(Y, \mathcal{Y})$ is called a plain texture if it satisfies any of the following equivalent conditions:

(1) $P_y \not\subseteq Q_y$ for all $y \in Y$
(2) $A = \bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ for all $A_i \in \mathcal{Y}, i \in I$

For a set $A \in \mathcal{Y}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap\Big\{ \bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{Y}, \quad A = \bigvee\{A_i \mid i \in I\}\Big\}.$$

Theorem 1 ([3]). *In any texture space $(Y, \mathcal{Y})$, the following statements hold:*

1. $y \notin A \Rightarrow A \subseteq Q_y \Rightarrow y \notin A^\flat$ *for all* $y \in Y$, $A \in \mathcal{Y}$.
2. $A^\flat = \{y \mid A \not\subseteq Q_y\}$ *for all* $A \in \mathcal{Y}$.
3. *For* $A_j \in \mathcal{Y}$, $j \in J$ *we have* $(\bigvee_{j\in J} A_j)^\flat = \bigcup_{j\in J} A_j^\flat$.
4. A *is the smallest element of* $\mathcal{Y}$ *containing* $A^\flat$ *for all* $A \in \mathcal{Y}$.
5. *For* $A, B \in \mathcal{Y}$, *if* $A \not\subseteq B$ *then there exists* $y \in Y$ *with* $A \not\subseteq Q_y$ *and* $P_y \not\subseteq B$.
6. $A = \bigcap\{Q_y \mid P_y \not\subseteq A\}$ *for all* $A \in \mathcal{Y}$.
7. $A = \bigvee\{P_y \mid A \not\subseteq Q_y\}$ *for all* $A \in \mathcal{Y}$.

Example 1 ([3]). (1) If $\mathcal{P}(X)$ is the powerset of a set X, then $(X, \mathcal{P}(X))$ is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$. The mapping $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is a complementation on the texture $(X, \mathcal{P}(X))$.
(2) Setting $\mathbb{I} = [0, 1]$, $\mathcal{J} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ gives the unit interval texture $(\mathbb{I}, \mathcal{J})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$. And the mapping $\iota : \mathcal{J} \to \mathcal{J}$, $\iota[0, r] = [0, 1 - r)$, $\iota[0, r) = [0, 1 - r]$ is a complementation on this texture.
(3) The texture $(L, \mathcal{L}, \lambda)$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$, $\lambda((0, r]) = (0, 1 - r]$. For $r \in L$, $P_r = (0, r] = Q_r$.
(4) $\mathcal{Y} = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, Y\}$ is a simple texturing of $Y = \{a, b, c\}$. $P_a = \{a, b\}$, $P_b = \{b\}$, $P_c = \{b, c\}$. It is not possible to define a complementation on $(Y, \mathcal{Y})$.
(5) If $(Y, \mathcal{Y}), (Z, \mathcal{Z})$ are textures, the product texturing $\mathcal{Y} \otimes \mathcal{Z}$ of $Y \times Z$ consists of arbitrary intersections of sets of the form $(A \times Z) \cup (Y \times B)$, $A \in \mathcal{Y}, B \in \mathcal{Z}$, and $(Y \times Z, \mathcal{Y} \otimes \mathcal{Z})$ is called the product of $(Y, \mathcal{Y})$ and $(Z, \mathcal{Z})$. For $y \in Y, z \in Z$, $P_{(y,z)} = P_y \times P_z$ and $Q_{(y,z)} = (Q_y \times Z) \cup (Y \times Q_z)$.

Definition 1 ([3]). Let $(Y, \mathcal{Y})$ and $(Z, \mathcal{Z})$ be textures. Then

1. $r \in \mathcal{P}(Y) \otimes \mathcal{Z}$ is called a relation on $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$ if it satisfies
   - (R1) $r \not\subseteq \overline{Q}(y, z)$, $P_{y'} \not\subseteq Q_y \Rightarrow r \not\subseteq \overline{Q}(y', z)$.
   - (R2) $r \not\subseteq \overline{Q}(y, z) \Rightarrow \exists y' \in Y$ such that $P_y \not\subseteq Q_{y'}$ and $r \not\subseteq \overline{Q}(y', z)$.
2. $R \in \mathcal{P}(Y) \otimes \mathcal{Z}$ is called a co-relation on $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$ if it satisfies
   - (CR1) $\overline{P}(y, z) \not\subseteq R$, $P_y \not\subseteq Q_{y'} \Rightarrow \overline{P}(y', z) \not\subseteq R$.
   - (CR2) $\overline{P}(y, z) \not\subseteq R \Rightarrow \exists y' \in Y$ such that $P_{y'} \not\subseteq Q_y$ and $\overline{P}(y', z) \not\subseteq R$.
3. A pair (r, R), where r is a relation and R a co-relation on $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$ is called a direlation on $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$.

For a texture $(Y, \mathcal{Y})$ the identity direlation $(i_{(Y,\mathcal{Y})}, I_{(Y,\mathcal{Y})})$ is defined by $i_{(Y,\mathcal{Y})} = \bigvee\{\overline{P}(y, y) \mid y \in Y\}$ and $I_{(Y,\mathcal{Y})} = \bigcap\{\overline{Q}(y, y) \mid y \in Y^\flat\}$.

For $A \subseteq Y$, $r^{\to}A = \bigcap\{Q_z \mid \forall y, r \not\subseteq \overline{Q}_{(y,z)} \Rightarrow A \subseteq Q_y\}$ is called the A-section of r and $R^{\to}A = \bigvee\{P_z \mid \forall y, \overline{P}_{(y,z)} \not\subseteq R \Rightarrow P_y \subseteq A\}$ is called the A-section of R.

For $B \subseteq Z$, $r^{\leftarrow}B = \bigvee\{P_y \mid \forall z, r \not\subseteq \overline{Q}_{(y,z)} \Rightarrow P_z \subseteq B\}$ is called the B-presection of r and $R^{\leftarrow}B = \bigcap\{Q_y \mid \forall z, \overline{P}_{(y,z)} \not\subseteq R \Rightarrow B \subseteq Q_z\}$ is called the B-presection of R.

Proposition 1 ([3]). *If (r, R) is a direlation on $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$ then $r^{\rightarrow}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} r^{\rightarrow}A_i$, $R^{\rightarrow}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} R^{\rightarrow}A_i$, $r^{\leftarrow}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} r^{\leftarrow}B_j$ and $R^{\leftarrow}(\bigvee_{j\in J} B_j) = \bigvee_{j\in J} R^{\leftarrow}B_j$ for any $A_i \in \mathcal{Y}$, $B_j \in \mathcal{Z}$, $i \in I$, $j \in J$.*

Definition 2 ([3]). A direlation (f, F) from $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$, is called a difunction from $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$ if it satisfies the following two conditions:

(DF1) For $y, y' \in Y$, $P_y \not\subseteq Q_{y'} \Rightarrow \exists z \in Z$ with $f \not\subseteq \overline{Q}_{(y,z)}$ and $\overline{P}_{(y',z)} \not\subseteq F$.

(DF2) For $z, z' \in Z$ and $y \in Y$, $f \not\subseteq \overline{Q}_{(y,z)}$ and $\overline{P}_{(y,z')} \not\subseteq F \Rightarrow P_{z'} \not\subseteq Q_z$.

(f, F) is called surjective if $\forall z, z' \in Z$ $P_z \not\subseteq Q_{z'} \Rightarrow \exists y \in Y$ with $f \not\subseteq \overline{Q}_{(y,z')}$ and $\overline{P}_{(y,z)} \not\subseteq F$. (f, F) is called injective if $\forall y, y' \in Y$, $z \in Z$ ($f \not\subseteq \overline{Q}_{(y,z)}$ and $\overline{P}_{(y',z)} \not\subseteq F) \Rightarrow P_y \not\subseteq Q_{y'}$.

In particular, the identity direlation (i_Y, I_Y) is a difunction on $(Y, \mathcal{Y})$.

Proposition 2 ([3]). *For a difunction (f, F) from $(Y, \mathcal{Y})$ to $(Z, \mathcal{Z})$, the following properties are satisfied:*

1. *$f^{\leftarrow}B = F^{\leftarrow}B$ for each $B \in \mathcal{Z}$.*
2. *$f^{\leftarrow}\emptyset = F^{\leftarrow}\emptyset = \emptyset$ and $f^{\leftarrow}Z = F^{\leftarrow}Z = Y$.*
3. *$A \subseteq F^{\leftarrow}(f^{\rightarrow}A)$ and $f^{\rightarrow}(F^{\leftarrow}B) \subseteq B$ for all $A \in \mathcal{Y}$, $B \in \mathcal{Z}$.*
4. *If (f, F) is surjective then $F^{\rightarrow}(f^{\leftarrow}B) = B = f^{\rightarrow}(F^{\leftarrow}B)$ for all $B \in \mathcal{Z}$.*
5. *If (f, F) is injective then $F^{\leftarrow}(f^{\rightarrow}A) = A = f^{\leftarrow}(F^{\rightarrow}A)$ for all $A \in \mathcal{Y}$.*

A ditopology on a texture $(Y, \mathcal{Y})$ is a pair (τ, κ), where $\tau, \kappa \subseteq \mathcal{Y}$ and the set of open sets τ satisfies

(T_1) $S, \emptyset \in \tau$

(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$

(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set of closed sets κ satisfies

(CT_1) $S, \emptyset \in \kappa$

(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$

(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

In this case, $(Y, \mathcal{Y}, \tau, \kappa)$ is called a ditopological texture space. So a ditopology can be considered as a "topology" in which there is no need for a relation between the open and closed sets to exist [1].

Let $(Y, \mathcal{Y})$ be a texture and $D \subseteq Y$ with $D \neq \emptyset$. $\{A, B\} \subseteq \mathcal{P}(Y)$ is called a partition of D if $A \cap D \neq \emptyset$, $D \not\subseteq B$ and $A \cap D = B \cap D$ [9].

Let $(Y, \mathcal{Y}, \tau, \kappa)$ be a ditopological texture space and $D \subseteq Y$. D is called connected if there is no partition $\{A, B\}$ with $A \in \tau$ and $B \in \kappa$ [9].

Graded Ditopological Texture Spaces [6]. Consider two textures $(Y, \mathcal{Y})$ and $(Z, \mathcal{Z})$. A graded ditopological texture space is a tuple $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ where the mappings $\mathcal{T}, \mathcal{K} : \mathcal{Y} \to \mathcal{Z}$ satisfy following conditions:

(GT_1) $\mathcal{T}(Y) = \mathcal{T}(\emptyset) = Z$
(GT_2) $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)$ $\forall A_1, A_2 \in \mathcal{Y}$
(GT_3) $\bigcap_{j \in J} \mathcal{T}(A_j) \subseteq \mathcal{T}(\bigvee_{j \in J} A_j)$ $\forall A_j \in \mathcal{Y}, j \in J$
(GCT_1) $\mathcal{K}(Y) = \mathcal{K}(\emptyset) = Z$
(GCT_2) $\mathcal{K}(A_1) \cap \mathcal{K}(A_2) \subseteq \mathcal{K}(A_1 \cup A_2)$ $\forall A_1, A_2 \in \mathcal{Y}$
(GCT_3) $\bigcap_{j \in J} \mathcal{K}(A_j) \subseteq \mathcal{K}(\bigcap_{j \in J} A_j)$ $\forall A_j \in \mathcal{Y}, j \in J$.

In this case $\mathcal{T}$ is called a $(Z, \mathcal{Z})$-graded topology and $\mathcal{K}$ a $(Z, \mathcal{Z})$-graded cotopology on $(Y, \mathcal{Y})$. For $z \in Z$ it is defined that

$$\mathcal{T}^z = \{A \in \mathcal{Y} \mid P_z \subseteq \mathcal{T}(A)\}, \; \mathcal{K}^z = \{A \in \mathcal{Y} \mid P_z \subseteq \mathcal{K}(A)\}.$$

So $(\mathcal{T}^z, \mathcal{K}^z)$ is a ditopology on $(Y, \mathcal{Y})$ for each $z \in Z$. Namely, if $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ is a graded ditopological texture space, then there exists a ditopological texture space $(Y, \mathcal{Y}, \mathcal{T}^z, \mathcal{K}^z)$ for each $z \in Z$.

Let $(Y_k, \mathcal{Y}_k, \mathcal{T}_k, \mathcal{K}_k, Z_k, \mathcal{Z}_k)$, $k = 1, 2$ be graded ditopological texture spaces, $(f, F) : (Y_1, \mathcal{Y}_1) \to (Y_2, \mathcal{Y}_2)$, $(h, H) : (Z_1, \mathcal{Z}_1) \to (Z_2, \mathcal{Z}_2)$ difunctions. For the pair $((f, F), (h, H))$, (f, F) is called continuous w.r.t. (h, H) if $H^{\leftarrow}\mathcal{T}_2(A) \subseteq \mathcal{T}_1(F^{\leftarrow}A)$ $\forall A \in \mathcal{Y}_2$, and cocontinuous w.r.t. (h, H) if $h^{\leftarrow}\mathcal{K}_2(A) \subseteq \mathcal{K}_1(f^{\leftarrow}A)$ $\forall A \in \mathcal{Y}_2$. If (f, F) is continuous and cocontinuous w.r.t. (h, H) then it is said to be a bicontinuous difunction w.r.t. (h, H).

Example 2 ([6])**.** Consider the discrete texture $(Z, \mathcal{Z}) = (1, \mathcal{P}(1))$ (The notation 1 denotes the set $\{0\}$) and takes a ditopological texture space $(Y, \mathcal{Y}, \tau, \kappa)$. Then the mappings $\tau^g, \kappa^g : \mathcal{Y} \to \mathcal{P}(1)$ defined by $\tau^g(A) = 1 \Leftrightarrow A \in \tau$ and $\kappa^g(A) = 1 \Leftrightarrow A \in \kappa$ form a graded ditopological texture space $(Y, \mathcal{Y}, \tau^g, \kappa^g, Z, \mathcal{Z})$. In this case (τ^g, κ^g) is called a graded ditopology on $(Y, \mathcal{Y})$ corresponding to ditopology (τ, κ). Thus graded ditopological texture spaces are more general than ditopological texture spaces.

2. Main results

Definition 3. Let $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ be a graded ditopological texture space The function $\mathfrak{Con} : \mathcal{P}(Y) \to \mathcal{Z}$ defined by

$$\mathfrak{Con}(D) = \bigcap \{Q_z \mid \exists A \in \mathcal{T}^z \; \exists B \in \mathcal{K}^z \; : \; \{A, B\} \text{ is a partition of } D\}$$

for all $D \in \mathcal{P}(Y)$, is called connectedness function of the graded ditopological texture space $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$. $\mathfrak{Con}(D)$ is called the grade of connectedness of D.

Proposition 3. *Let $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ be a graded ditopological texture space where the texture $(Z, \mathcal{Z})$ is plain. Then we have*

$$\mathfrak{Con}(Y) = Z \Leftrightarrow \forall A \in \mathcal{Y} \setminus \{\emptyset, Y\} \;\; \mathcal{T}(A) \cap \mathcal{K}(A) = \emptyset.$$

Proof. $(\Rightarrow)$: Suppose that there exists a set $D \in \mathcal{Y}\setminus\{\emptyset, Y\}$ such that $\mathcal{T}(D)\cap \mathcal{K}(D) \neq \emptyset$. So there exists an element $z \in Z$ such that $P_z \subseteq \mathcal{T}(D) \cap \mathcal{K}(D)$ and this implies $D \in \mathcal{T}^z$, $D \in \mathcal{K}^z$. Besides, $\{D, D\}$ is a partition of Y because $D \neq Y$, $D \neq \emptyset$, $D \cap Y \neq \emptyset$ and $Y \not\subseteq D$. Thus we have

$$\mathfrak{Con}(Y) = \bigcap\{Q_t \mid \exists A \in \mathcal{T}^t\ \exists B \in \mathcal{K}^t\ :\ \{A, B\} \text{ is a partition of } Y\} \subseteq Q_z.$$

Since $(Z, \mathcal{Z})$ is plain we have $z \notin Q_z$ and so $Z \not\subseteq Q_z$. By considering $\mathfrak{Con}(Y) \subseteq Q_z$ and $Z \not\subseteq Q_z$, we get $\mathfrak{Con}(Y) \neq Z$.
$(\Leftarrow)$: Suppose that $\mathfrak{Con}(Y) \neq Z$. So we have

$$\exists t \in Z\ \exists A \in \mathcal{T}^t\ \exists B \in \mathcal{K}^t\ :\ \{A, B\} \text{ is a partition of } Y$$

because otherwise, it would be $\mathfrak{Con}(Y) = Z$ by Definition 3. Also ,since $\{A, B\}$ is a partition of Y, we have $A = B$, $A \neq \emptyset$, $A \neq Y$. By considering $A \in \mathcal{T}^t$ and $B \in \mathcal{K}^t$ we get $P_t \subseteq \mathcal{T}(A) \cap \mathcal{K}(A)$ and so $\mathcal{T}(A) \cap \mathcal{K}(A) \neq \emptyset$. □

Proposition 4. *Let $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ be a graded ditopological texture space where the texture $(Z, \mathcal{Z})$ is plain. If for every $a, b \in Y$ with $a \neq b$ there exists a set $D \in \mathcal{P}(Y)$ such that $a, b \in D$ and $P_z \subseteq \mathfrak{Con}(D)$ then $P_z \subseteq \mathfrak{Con}(Y)$.*

Proof. Let the hypotesis be satisfied. Assume that $P_z \not\subseteq \mathfrak{Con}(Y)$. Then we have

$$\exists t \in Z\ (\exists A \in \mathcal{T}^t\ \exists B \in \mathcal{K}^t\ :\ P_z \not\subseteq Q_t \text{ and } \{A, B\} \text{ is a partition of } Y)$$

by Definition 3. Since $\{A, B\}$ is a partition of Y, we have $A = B$, $A \neq \emptyset$ and $A \neq Y$. So, if we take $a \in A$ and $b \in Y \setminus A$, then we have

$$\exists D \subseteq Y\ : a, b \in D \text{ and } P_z \subseteq \mathfrak{Con}(D)$$

by the hypotesis. Also since $P_z \not\subseteq Q_t$ we get $P_t \subseteq P_z \subseteq \mathfrak{Con}(D)$. Considering $A \cap D \neq \emptyset$ and $D \not\subseteq A$ we conclude that $\{A, A\}$ is a partition of D. Since $A \in \mathcal{T}^t$ and $A = B \in \mathcal{T}^t$, we get $\mathfrak{Con}(D) \subseteq Q_t$. Since $P_t \subseteq \mathfrak{Con}(D)$ and $\mathfrak{Con}(D) \subseteq Q_t$, we get $P_t \subseteq Q_t$. However, this gives a contradiction because the texture $(Z, \mathcal{Z})$ is plain. □

Theorem 2. *Let $(Y_k, \mathcal{Y}_k, \mathcal{T}_k, \mathcal{K}_k, Z_k, \mathcal{Z}_k)_{k=1,2}$ be graded ditopological texture spaces such that $(Z_1, \mathcal{Z}_1)$ and $(Z_2, \mathcal{Z}_2)$ are plain. Let $(f, F) : (Y_1, \mathcal{Y}_1) \to (Y_2, \mathcal{Y}_2)$, $(g, G) : (Z_1, \mathcal{Z}_1) \to (Z_2, \mathcal{Z}_2)$ be surjective difunctions. If (f, F) is bicontinuous w.r.t. (g, G) then*

$$\mathfrak{Con}(Y_1) = Z_1 \Rightarrow \mathfrak{Con}(Y_2) = Z_2.$$

Proof. Let $\mathfrak{Con}(Y_1) = Z_1$ and suppose that $\mathfrak{Con}(Y_2) \neq Z_2$. Then, we have $\mathcal{T}_2(A) \cap \mathcal{K}_2(A) \neq \emptyset$ for some $A \in \mathcal{Y}_2 \setminus \{\emptyset, Y_2\}$ by Proposition 3. Therefore, we get

$$\exists z_2 \in Z_2\ :\ P_{z_2} \subseteq \mathcal{T}_2(A) \text{ and } P_{z_2} \subseteq \mathcal{K}_2(A).$$

Since (f, F) is bicontinuous w.r.t. (g, G) and (g, G) is surjective we have

$$\emptyset \neq g^{\leftarrow}(P_{z_2}) \subseteq g^{\leftarrow}(\mathcal{T}_2(A)) \subseteq \mathcal{T}_1(f^{\leftarrow}(A))$$

and

$$\emptyset \neq g^{\leftarrow}(P_{z_2}) \subseteq g^{\leftarrow}(\mathcal{K}_2(A)) \subseteq \mathcal{K}_1(f^{\leftarrow}(A)).$$

That is, $\mathcal{T}_1(f^{\leftarrow}(A)) \cap \mathcal{K}_1(f^{\leftarrow}(A)) \neq \emptyset$ and $f^{\leftarrow}(A) \in \mathcal{Y}_1$. Also, since (f, F) is surjective, $A \neq \emptyset$ and $A \neq Y_2$ we get $f^{\leftarrow}(A) \neq \emptyset$ and $f^{\leftarrow}(A) \neq Y_1$. Therefore using Proposition 3, we obtain that $\mathfrak{Con}(Y_1) \neq Z_1$ which contradicts with $\mathfrak{Con}(Y_1) = Z_1$. Hence, we get $\mathfrak{Con}(Y_2) = Z_2$. □

Definition 4. Let $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ be a graded ditopological texture space and $D \subseteq Y$. The family defined by

$$S_{\mathfrak{Con}}(D) = \{P_z \mid z \notin \{t \in Z \mid \exists A \in \mathcal{T}^t \, \exists B \in \mathcal{K}^t \ : \ \{A, B\} \text{ is a partition of } D\}\}$$

is called the connectedness spectrum of D.

Proposition 5. *Let $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ be a graded ditopological texture space where the texture $(Z, \mathcal{Z})$ is plain. Then, we have*

$$P_z \subseteq \mathfrak{Con}(D) \Rightarrow P_z \in S_{\mathfrak{Con}}(D)$$

for all $D \subseteq Y$.

Proof. Let $P_z \subseteq \mathfrak{Con}(D)$ and suppose that $P_z \notin S_{\mathfrak{Con}}(D)$. Thath way, we have a partition $\{A, B\}$ of D for some $A \in \mathcal{T}^z$ and $B \in \mathcal{K}^z$. This implies $\mathfrak{Con}(D) \subseteq Q_z$. Since $P_z \subseteq \mathfrak{Con}(D)$ we get $P_z \subseteq Q_z$. However, this gives a contradiction, because the texture $(Z, \mathcal{Z})$ is plain. □

Theorem 3. *Let $(Y_k, \mathcal{Y}_k, \mathcal{T}_k, \mathcal{K}_k, Z_k, \mathcal{Z}_k)_{k=1,2}$ be graded ditopological texture spaces. Let $(f, F) : (Y_1, \mathcal{Y}_1) \to (Y_2, \mathcal{Y}_2)$, $(g, G) : (Z_1, \mathcal{Z}_1) \to (Z_2, \mathcal{Z}_2)$ be difunctions. If (f, F) is bijective and bicontinuous w.r.t. (g, G) then*

$$P_{z_1} \in S_{\mathfrak{Con}_1}(D) \Rightarrow P_{z_2} \in S_{\mathfrak{Con}_2}(F^{\rightarrow}D)$$

for all $D \in \mathcal{P}(Y_1)$, where $P_{z_1} \in \mathcal{Z}_1$, $P_{z_2} \in \mathcal{Z}_2$ with $P_{z_1} \subseteq g^{\leftarrow}P_{z_2}$.

Proof. $(\Rightarrow)$: Suppose that $P_{z_2} \notin S_{\mathfrak{Con}_2}(F^{\rightarrow}D)$. Hence, we have

$$\exists A \in \mathcal{T}_2^{z_2} \ \exists B \in \mathcal{K}_2^{z_2} \ : \ \{A, B\} \text{ is a partition of } F^{\rightarrow}D.$$

Since $A \in \mathcal{T}_2^{z_2}$ and $B \in \mathcal{K}_2^{z_2}$ we get $P_{z_2} \subseteq \mathcal{T}_2(A)$ and $P_{z_2} \subseteq \mathcal{K}_2(B)$. Using the bicontinuity of (f, F) w.r.t. (g, G) we obtain

$$P_{z_1} \subseteq g^{\leftarrow}(P_{z_2}) = G^{\leftarrow}(P_{z_2}) \subseteq G^{\leftarrow}(\mathcal{T}_2(A)) \subseteq \mathcal{T}_1(F^{\leftarrow}A)$$

and

$$P_{z_1} \subseteq g^{\leftarrow}(P_{z_2}) \subseteq g^{\leftarrow}(\mathcal{K}_2(B)) \subseteq \mathcal{K}_1(f^{\leftarrow}B).$$

That means $F^{\leftarrow}A \in \mathcal{T}_1^{z_1}$ and $f^{\leftarrow}B \in \mathcal{K}_1^{z_1}$.

Now, since $\{A, B\}$ is a partition of $F^{\rightarrow}D$, we have $A \cap F^{\rightarrow}D \neq \emptyset$, $F^{\rightarrow}D \not\subseteq B$ and $A \cap F^{\rightarrow}D = B \cap F^{\rightarrow}D$. Considering $A \cap F^{\rightarrow}D \neq \emptyset$, Proposition 1 and Proposition 2, we get

$$\emptyset \neq f^{\leftarrow}(A \cap F^{\rightarrow}D) = (f^{\leftarrow}A) \cap f^{\leftarrow}(F^{\rightarrow}D) \subseteq f^{\leftarrow}(A) \cap D$$

and thus $f^{\leftarrow}(A) \cap D \neq \emptyset$.

Since $F^{\to}D \not\subseteq B$, we have $F^{\to}D \not\subseteq Q_t$ and $P_t \not\subseteq B$ for some $t \in Z_2$ by Theorem 1 (5). This implies $P_t \subseteq F^{\to}D$ and $P_t \not\subseteq B$. Also $P_t \subseteq F^{\to}D$ implies $f^{\leftarrow}P_t \subseteq f^{\leftarrow}(F^{\to}D) \subseteq D$. On the other hand, we have $f^{\leftarrow}P_t \not\subseteq f^{\leftarrow}B$ because otherwise, $f^{\leftarrow}P_t \subseteq f^{\leftarrow}B$ would imply $P_t \subseteq F^{\to}(f^{\leftarrow}P_t) \subseteq F^{\to}(f^{\leftarrow}B) = B$ (since (f,F) is surjective) and so $P_t \subseteq B$, which contradicts the fact $P_t \not\subseteq B$. Thus, we have $f^{\leftarrow}P_t \subseteq D$ and $f^{\leftarrow}P_t \not\subseteq f^{\leftarrow}B$ and therefore $D \not\subseteq f^{\leftarrow}B$.

Since (f,F) is injective, we have $f^{\leftarrow}(F^{\to}D) = D$. So, considering $A \cap F^{\to}D = B \cap F^{\to}D$ we get

$$f^{\leftarrow}(A \cap F^{\to}D) = f^{\leftarrow}(B \cap F^{\to}D) \Rightarrow f^{\leftarrow}A \cap f^{\leftarrow}(F^{\to}D) = f^{\leftarrow}B \cap f^{\leftarrow}(F^{\to}D)$$
$$\Rightarrow f^{\leftarrow}A \cap D = f^{\leftarrow}B \cap D.$$

Hence, $\{f^{\leftarrow}A,\ f^{\leftarrow}A\}$ is a partition of D and so $P_{z_1} \notin S_{\mathfrak{Con}_1}(D)$. $\square$

Example 3. Let $(Y, \mathcal{Y}, \tau, \kappa)$ be a ditopological texture space and $(Y, \mathcal{Y}, \tau^g, \kappa^g, Z, \mathcal{Z})$ be the graded ditopological texture space corresponding to $(Y, \mathcal{Y}, \tau, \kappa)$. That is, $(Z, \mathcal{Z}) = (1 = \{0\}, \mathcal{P}(1) = \{\emptyset, 1\})$ is the discrete texture on a singleton and

$$\tau^g(A) = \begin{cases} 1, & A \in \tau; \\ \emptyset, & A \notin \tau; \end{cases}$$

$$\kappa^g(A) = \begin{cases} 1, & A \in \kappa; \\ \emptyset, & A \notin \kappa; \end{cases}$$

for all $A \in \mathcal{Y}$. In this case, a set $D \in \mathcal{Y}$ is connected in ditopological texture space $(Y, \mathcal{Y}, \tau, \kappa)$ if and only if $\mathfrak{Con}(D) = Z$ in graded ditopological texture space $(Y, \mathcal{Y}, \tau^g, \kappa^g, Z, \mathcal{Z})$.

Example 4. Let $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ be a graded ditopological texture space and $D \in \mathcal{Y}$. In this case,

$$D \text{ is connected in ditopological texture space } (Y, \mathcal{Y}, \mathcal{T}^z, \mathcal{K}^z) \Leftrightarrow P_z \in S_{\mathfrak{Con}}(D)$$

for each $z \in Z$.

Example 5. Let $(Y, \mathcal{Y} = \mathcal{P}(Y))$ and $(Z, \mathcal{Z} = \mathcal{P}(Z))$ be discrete textures with $Z = \{1, 2, 3, 4\}$ where Y has more than one element. If we define $\mathcal{T}, \mathcal{K} : \mathcal{Y} \to \mathcal{Z}$ by

$$\mathcal{T}(A) = \begin{cases} Z, & A = \emptyset \text{ or } A = Y; \\ \{3\}, & \text{otherwise}; \end{cases}$$

$$\mathcal{K}(A) = \begin{cases} Z, & A = \emptyset \text{ or } A = Y; \\ \{3, 4\}, & \text{otherwise}; \end{cases}$$

for all $A \in \mathcal{Y}$, then we have a graded ditopological texture space $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$. Also we have $\mathcal{T}^3 = \mathcal{Y} = \mathcal{P}(S)$, $\mathcal{T}^1 = \mathcal{T}^2 = \mathcal{T}^4 = \{Y, \emptyset\}$, $\mathcal{K}^3 = \mathcal{K}^4 = \mathcal{Y} = \mathcal{P}(S)$, $\mathcal{K}^1 = \mathcal{K}^2 = \{Y, \emptyset\}$ since $P_1 = \{1\}$, $P_2 = \{2\}$, $P_3 = \{3\}$, and $P_4 = \{4\}$.

Now, let $D \in \mathcal{Y}$ be a set which has more than one element. If we define $A = \{d\}$ for an element $d \in D$, then $\{A, A\}$ is a partition of D. Also we have $A \in \mathcal{T}^3$ and $A \in \mathcal{K}^3$. Hence, we get $\mathfrak{Con}(D) = Q_3 = Z \setminus \{3\} = \{1, 2, 4\}$ and $S_{\mathfrak{Con}}(D) = \{P_1, P_2, P_4\}$.

3. Conclusion

In this paper, two different types of connectedness notions for graded ditopological texture spaces are introduced in accordance with the connectedness notion in ditopological texture spaces given in [9]. Firstly, we present the concept of connectedness function (see Definition 3) which gives the grade of connectedness of a set. Afterwards, through spectral theory, connectedness spectrum of a set is given (see Definition 4). Also, the properties of these connectedness notions and their relationships with the connectedness notion in ditopological case are investigated.

The concept of connectedness function is stronger than the concept of connectedness spectrum (see Proposition 5). However, it has some disadvantages in the theory of graded ditopological texture spaces. For instance, generalizations of several properties in ditopological case to the graded ditopological case are valid if the texture $(Z, \mathcal{Z})$ is plain for the graded ditopological texture space $(Y, \mathcal{Y}, \mathcal{T}, \mathcal{K}, Z, \mathcal{Z})$ (see Proposition 3 and 4, Theorem 2). On the other hand, the concept of connectedness spectrum works better than the concept of connectedness function (see Theorem 3). However, the concept of connectedness function is smoother than the concept of connectedness spectrum.

Building connectedness setting in the theory of graded ditopological texture spaces can be beneficial to study in this theory. It can also serve to discover new properties in the theory. Considering the interrelations among the structures graded ditopological texture spaces, ditopological texture spaces, fuzzy topological spaces, nearness structure on texture spaces, this research has the potential to improve the related study fields.

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